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Research Article

An Optimized hybrid one step method for numerical integration of second order initial value problems

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Abstract

This article presents an optimized hybrid one-step block method for the direct numerical integration of second-order initial value problems (IVPs) of the form $y'' = f(x,y,y')$. The method is derived using a power series basis function through a collocation and interpolation technique, allowing for direct integration without the need to reduce the second-order equation into a system of first-order ordinary differential equations. To maximize the accuracy of the scheme, the off-grid collocation points are optimized by analyzing the local truncation error (LTE). By forcing the leading terms of the error expansion to zero, we determined the optimal positions for the hybrid points at $c = \frac{1}{2} - \frac{\sqrt{21}}{14}$ and $b = \frac{1}{2} + \frac{\sqrt{21}}{14}$. The resulting hybrid block method achieves a high local truncation error of $O(h^8)$, ensuring superior precision and rapid convergence. The continuous formulation of the method facilitates the simultaneous generation of approximate solutions at both grid and off-grid nodes. The consistency and efficiency of the proposed scheme make it a robust computational tool for solving dynamical problems in fields such as physics, engineering, and biology.

1. Introduction

Initial value problems (IVPs) represent a cornerstone of mathematical and physical modeling, focused on predicting a system's evolution based on a defined starting state. These models are essential across disciplines ranging from thermodynamics and mechanical oscillations to electrical engineering and economic forecasting. The complexity of these systems is typically categorized by the order of the governing differential equation [1]. For instance, a cooling liquid follows a first-order model, while mass-spring systems and RLC circuits represent second- and third-order dynamics, respectively.

Because analytical solutions are often unattainable, researchers rely on numerical approximation. Traditionally, second-order ordinary differential equations (ODEs) were solved via reduction of order, which transforms the equation into a system of first-order ODEs. However, this transition can introduce unnecessary computational complexity and heighten the risk of accumulated integration errors [2–5]. Conversely, direct methods bypass this transformation, integrating the second-order equation in its original form. This approach has seen a surge in interest [2–10]. Among these, one-step methods are frequently employed, as they calculate the subsequent state using only the data from the immediately preceding step [11–15].

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One-step implicit hybrid methods have emerged as a robust alternative. By merging the characteristics of implicit and explicit schemes, these methods seek a "best-of-both-worlds" scenario: Implicit components provide superior numerical stability but necessitate solving non-linear systems and Explicit components offer ease of implementation but are often constrained by stricter stability limits.

Hybrid frameworks aim to balance these trade-offs to optimize performance [12, 16, 17]. The effectiveness of these hybrid methods depends heavily on the strategic selection of interpolation and collocation points. Through precise parameter optimization, researchers can minimize local truncation errors, ensuring higher accuracy without a proportional increase in computational overhead [12, 17].

Current literature focuses on three primary pillars of improvement:

Refined Accuracy: Utilizing advanced collocation techniques to achieve higher-order precision, thereby reducing the number of steps required for a simulation [12, 17–22].
Robust Stability: Developing methods with enhanced zero-stability to ensure that numerical solutions remain bounded during long-term simulations [12, 17].
Algorithmic Efficiency: Streamlining implementations to reduce function evaluations and optimize linear solvers [12, 17–22]. Additionally, the scope of these methods is expanding into fuzzy differential equations, allowing for the integration of uncertainty into the modeling process [12, 16].

2. Mathematical Formulation

To solve the general second-order initial value problem (IVP) efficiently, we develop a continuous approximation. This approach avoids the complexity of order-reduction while maintaining high local accuracy.

We begin by considering the second-order ODE defined on the interval $[a,z]$:

$$y''(x) = f(x,y,y'); \quad y(a) = y_0, \quad y'(a) = y'_0 \tag{1}$$

The exact solution $y(x)$ is locally approximated by a polynomial $P(x)$ derived from a power series expansion of the form:

$$y(x) \approx P(x) = \sum_{n=0}^{m} a_n x^n \tag{2}$$

Where m is the degree of the polynomial, and a_n are the unknown real coefficients to be determined. To satisfy the differential equation(1), we obtain the first and second derivatives of our approximant:

$$P'(x) = \sum_{n=1}^{m} n a_n x^{n-1} \tag{3}$$

$$f(x) \approx P''(x) = \sum_{n=2}^{m} n(n-1) a_n x^{n-2} \tag{4}$$

To uniquely determine the coefficients a_n, we employ a hybrid strategy involving both interpolation and collocation points. The total number of points determines the polynomial degree such that $m = (I+C)-1$, where I is the number of interpolation points and C is the number of collocation points.

The coefficients are determined by the following constraints: Interpolation: The approximant $P(x)$ is interpolated at points x_{n+i} for $i \in \{0, \frac{1}{2}\}$, ensuring the solution passes through these specific nodes.

Collocation: The second derivative $P''(x)$ is equated to the function $f(x,y,y')$ at the collocation points x_{n+t} for $t \in \{0, c, \frac{1}{2}, b, 1\}$. The distribution of these nodes is strictly governed by $0 < c < \frac{1}{2} < b < 1$. This specific configuration allows us to include both grid and off-grid nodes, effectively increasing the method's order of accuracy without expanding the step size h. The idea is to approximate the exact solution $y(x)$ of (1) in the partition

$$\pi_{[a,z]} = [a = x_o < x_1 < x_2 < x_3 < \cdots x_n = z] \tag{5}$$

on the interval $[a,z]$ by the power series function of the form in equation (2).
Power series of the form (2) will be used as basis function to approximate the second order initial value problem of the form in (1). We approximate (1) using the power series below for the derivation of the hybrid one-step method $(m = C+I-1)$,

$$P(x) = \sum_{n=0}^{6} a_n x^n \tag{6}$$

$$y(x) = a_0x^0 + a_1x^1 + a_2x^2 + a_3x^3 + a_4x^4 + +a_5x^5 + +a_6x^6 \tag{7}$$

Differentiate (7) twice we have:

$$f(x) = y''(x) = 2a_2 + 6a_3x + 12a_4x^2 + 20a_5x^3 + 30a_6x^4 \tag{8}$$

Where $x \in [a,z]$, $a'_n s$ are real unknown parameters to be uniquely determined. Let the solution of equation (2) be sought on the partition:

$$\pi_\mu = [a = x_o < x_1 < x_2 < x_3 < \cdots x_N = z]$$

on the interval $[a,z]$ with a constant step size h, given by;

$$h = x_{n+1} - x_n, \quad n = 0, 1, 2, \cdots, N.$$

Interpolating equation (7) at $x = x_n$ and $x = x_{n+\frac{1}{2}}$, also collocating (8) at $x = x_n$, $x = x_{n+c}$, $x = x_{n+\frac{1}{2}}$, $x = x_{n+b}$ and $x = x_{n+1}$, we arrive at system of seven equations with 7 real unknown parameters $a_n's$, $n = 0(1)6$, that can be written in matrix form as:

$$\begin{pmatrix} 1 & x_n & x_n^2 & x_n^3 & x_n^4 & x_n^5 & x_n^6 \\ 1 & x_{n+\frac{1}{2}} & x_{n+\frac{1}{2}}^2 & x_{n+\frac{1}{2}}^3 & x_{n+\frac{1}{2}}^4 & x_{n+\frac{1}{2}}^5 & x_{n+\frac{1}{2}}^6 \\ 0 & 0 & 2 & 6x_n & 12x_n^2 & 20x_n^3 & 30x_n^4 \\ 0 & 0 & 2 & 6x_{n+c} & 12x_{n+c}^2 & 20x_{n+c}^3 & 30x_{n+c}^4 \\ 0 & 0 & 2 & 6x_{n+\frac{1}{2}} & 12x_{n+\frac{1}{2}}^2 & 20x_{n+\frac{1}{2}}^3 & 30x_{n+\frac{1}{2}}^4 \\ 0 & 0 & 2 & 6x_{n+b} & 12x_{n+b}^2 & 20x_{n+b}^3 & 30x_{n+b}^4 \\ 0 & 0 & 2 & 6x_{n+1} & 12x_{n+1}^2 & 20x_{n+1}^3 & 30x_{n+1}^4 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{pmatrix} = \begin{pmatrix} y_n \\ y_{n+\frac{1}{2}} \\ f_n \\ f_{n+c} \\ f_{n+\frac{1}{2}} \\ f_{n+b} \\ f_{n+1} \end{pmatrix}.$$

Solving the above system of equation for a_0, a_1, a_2, a_3, a_4, a_5 and a_6 and inserting $a_i's$ into equation (7) yields the continuous form of the hybrid one-step method:

$$y(x) = \alpha_0 y_n + \alpha_b y_{n+\frac{1}{2}} + h^2 \left[\beta_0 f_n + \beta_c f_{n+c} + \beta_{\frac{1}{2}} f_{n+\frac{1}{2}} + \beta_b f_{n+b} + \beta_1 f_{n+1} \right], \tag{9}$$

where

$$\alpha_0 = -\frac{x\left(7680b^4c^3 - 7680b^3c^4 - 11520b^4c^2 + 11520b^2c^4 + 3840b^4c + 13440b^3c^2 - 13440b^2c^3 - 3840bc^4 - 5760b^3c + 5760bc^3 + 1920b^2c - 1920bc^2\right)}{960b\left(2b^2 - 3b + 1\right)\left(2bc^2 - 2c^3 - 3bc + 3c^2 + b - c\right)c} + 1$$

$$\alpha_b = -\frac{x\left(-7680b^4c^3 + 7680b^3c^4 + 11520b^4c^2 - 11520b^2c^4 - 3840b^4c - 13440b^3c^2 + 13440b^2c^3 + 3840bc^4 + 5760b^3c - 5760bc^3 - 1920b^2c + 1920bc^2\right)}{960b\left(2b^2 - 3b + 1\right)\left(2bc^2 - 2c^3 - 3bc + 3c^2 + b - c\right)c}$$

$$\beta_0 = \frac{1}{15}\frac{x^6}{bc} - \frac{1}{20}\frac{x^5\left(8b^4c^2 - 8b^2c^4 - 12b^4c + 12bc^4 + 4b^4 - 4c^4 + 15b^2c - 15bc^2 - 7b^2 + 7c^2 + 3b - 3c\right)}{\left(4b^3c^2 - 4b^2c^3 - 6b^3c + 6bc^3 + 2b^3 + 7b^2c - 7bc^2 - 2c^3 - 3b^2 + 3c^2 + b - c\right)bc}$$

$$\frac{1}{12}\frac{x^4\left(8b^4c^3 - 8b^3c^4 - 14b^4c + 14bc^4 + 6b^4 + 15b^3c - 15bc^3 - 6c^4 - 7b^3 + 7c^3 + b - c\right)}{\left(4b^3c^2 - 4b^2c^3 - 6b^3c + 6bc^3 + 2b^3 + 7b^2c - 7bc^2 - 2c^3 - 3b^2 + 3c^2 + b - c\right)bc}$$

$$-\frac{1}{6}\frac{x^3\left(12b^4c^3 - 12b^3c^4 - 14b^4c^2 + 14b^2c^4 + 15b^3c^2 - 15b^2c^3 + 2b^4 - 2c^4 - 3b^3 + 3c^3 + b^2 - c^2\right)}{\left(4b^3c^2 - 4b^2c^3 - 6b^3c + 6bc^3 + 2b^3 + 7b^2c - 7bc^2 - 2c^3 - 3b^2 + 3c^2 + b - c\right)bc} + 1/2x^2$$

$$-\frac{1}{960b\left(2b^2 - 3b + 1\right)\left(2bc^2 - 2c^3 - 3bc + 3c^2 + b - c\right)c}($$

$$x(560b^4c^3 - 560b^3c^4 - 904b^4c^2 + 904b^2c^4 + 376b^4c + 1088b^3c^2 - 1088b^2c^3 - 376bc^4 - 32b^4 - 582b^3c + 582bc^3 + 32c^4 + 54b^3 + 209b^2c - 209bc^2 - 54c^3 - 25b^2 + 25c^2 + 3b - 3c))$$

$$\beta_c = \frac{1}{15}\frac{x^6\left(-2b^3 + 3b^2 - b\right)}{\left(4b^3c^2 - 4b^2c^3 - 6b^3c + 6bc^3 + 2b^3 + 7b^2c - 7bc^2 - 2c^3 - 3b^2 + 3c^2 + b - c\right)bc}$$

$$-\frac{1}{20}\frac{x^5\left(-4b^4 + 7b^2 - 3b\right)}{\left(4b^3c^2 - 4b^2c^3 - 6b^3c + 6bc^3 + 2b^3 + 7b^2c - 7bc^2 - 2c^3 - 3b^2 + 3c^2 + b - c\right)bc}$$

$$\frac{1}{12}\frac{x^4\left(-6b^4 + 7b^3 - b\right)}{\left(4b^3c^2 - 4b^2c^3 - 6b^3c + 6bc^3 + 2b^3 + 7b^2c - 7bc^2 - 2c^3 - 3b^2 + 3c^2 + b - c\right)bc}$$

$$-\frac{1}{6}\frac{x^3\left(-2b^4 + 3b^3 - b^2\right)}{\left(4b^3c^2 - 4b^2c^3 - 6b^3c + 6bc^3 + 2b^3 + 7b^2c - 7bc^2 - 2c^3 - 3b^2 + 3c^2 + b - c\right)bc}$$

$$-\frac{x\left(32b^4 - 54b^3 + 25b^2 - 3b\right)}{960b\left(2b^2 - 3b + 1\right)\left(2bc^2 - 2c^3 - 3bc + 3c^2 + b - c\right)c}$$

$$\beta_{\frac{1}{2}} = \frac{1}{15}\frac{x^6\left(-8b^3c^2 + 8b^2c^3 + 8b^3c - 8bc^3 - 8b^2c + 8bc^2\right)}{\left(4b^3c^2 - 4b^2c^3 - 6b^3c + 6bc^3 + 2b^3 + 7b^2c - 7bc^2 - 2c^3 - 3b^2 + 3c^2 + b - c\right)bc}$$

$$-\frac{1}{20}\frac{x^5\left(-16b^4c^2 + 16b^2c^4 + 16b^4c - 16bc^4 - 16b^2c + 16bc^2\right)}{\left(4b^3c^2 - 4b^2c^3 - 6b^3c + 6bc^3 + 2b^3 + 7b^2c - 7bc^2 - 2c^3 - 3b^2 + 3c^2 + b - c\right)bc}$$

$$\frac{1}{12}\frac{x^4\left(-16b^4c^3 + 16b^3c^4 + 16b^4c - 16bc^4 - 16b^3c + 16bc^3\right)}{\left(4b^3c^2 - 4b^2c^3 - 6b^3c + 6bc^3 + 2b^3 + 7b^2c - 7bc^2 - 2c^3 - 3b^2 + 3c^2 + b - c\right)bc}$$

$$-\frac{1}{6}\frac{x^3\left(-16b^4c^3 + 16b^3c^4 + 16b^4c^2 - 16b^2c^4 - 16b^3c^2 + 16b^2c^3\right)}{\left(4b^3c^2 - 4b^2c^3 - 6b^3c + 6bc^3 + 2b^3 + 7b^2c - 7bc^2 - 2c^3 - 3b^2 + 3c^2 + b - c\right)bc}$$

$$-\frac{x\left(480b^4c^3 - 480b^3c^4 - 592b^4c^2 + 592b^2c^4 + 112b^4c + 624b^3c^2 - 624b^2c^3 - 112bc^4 - 144b^3c + 144bc^3 + 32b^2c - 32bc^2\right)}{960b\left(2b^2 - 3b + 1\right)\left(2bc^2 - 2c^3 - 3bc + 3c^2 + b - c\right)c}$$

$$\beta_b = \frac{1}{15}\frac{x^6\left(2c^3 - 3c^2 + c\right)}{\left(4b^3c^2 - 4b^2c^3 - 6b^3c + 6bc^3 + 2b^3 + 7b^2c - 7bc^2 - 2c^3 - 3b^2 + 3c^2 + b - c\right)bc}$$

$$-1/20\frac{x^5\left(4c^4 - 7c^2 + 3c\right)}{\left(4b^3c^2 - 4b^2c^3 - 6b^3c + 6bc^3 + 2b^3 + 7b^2c - 7bc^2 - 2c^3 - 3b^2 + 3c^2 + b - c\right)bc}$$

$$\frac{1}{12}\,\frac{x^4\left(6c^4-7c^3+c\right)}{\left(4b^3c^2-4b^2c^3-6b^3c+6bc^3+2b^3+7b^2c-7bc^2-2c^3-3b^2+3c^2+b-c\right)bc}$$

$$-\frac{1}{6}\,\frac{x^3\left(2c^4-3c^3+c^2\right)}{\left(4b^3c^2-4b^2c^3-6b^3c+6bc^3+2b^3+7b^2c-7bc^2-2c^3-3b^2+3c^2+b-c\right)bc}$$

$$-\frac{x\left(-32c^4+54c^3-25c^2+3c\right)}{960b\left(2b^2-3b+1\right)\left(2bc^2-2c^3-3bc+3c^2+b-c\right)c}$$

$$\beta_1=\frac{1}{15}\,\frac{x^6\left(4b^3c^2-4b^2c^3-2b^3c+2bc^3+b^2c-bc^2\right)}{\left(4b^3c^2-4b^2c^3-6b^3c+6bc^3+2b^3+7b^2c-7bc^2-2c^3-3b^2+3c^2+b-c\right)bc}$$

$$-\frac{1}{20}\,\frac{x^5\left(8b^4c^2-8b^2c^4-4b^4c+4bc^4+b^2c-bc^2\right)}{\left(4b^3c^2-4b^2c^3-6b^3c+6bc^3+2b^3+7b^2c-7bc^2-2c^3-3b^2+3c^2+b-c\right)bc}$$

$$\frac{1}{12}\,\frac{x^4\left(8b^4c^3-8b^3c^4-2b^4c+2bc^4+b^3c-bc^3\right)}{\left(4b^3c^2-4b^2c^3-6b^3c+6bc^3+2b^3+7b^2c-7bc^2-2c^3-3b^2+3c^2+b-c\right)bc}$$

$$-\frac{1}{6}\,\frac{x^3\left(4b^4c^3-4b^3c^4-2b^4c^2+2b^2c^4+b^3c^2-b^2c^3\right)}{\left(4b^3c^2-4b^2c^3-6b^3c+6bc^3+2b^3+7b^2c-7bc^2-2c^3-3b^2+3c^2+b-c\right)bc}$$

$$-\frac{x\left(-80b^4c^3+80b^3c^4+56b^4c^2-56b^2c^4-8b^4c-32b^3c^2+32b^2c^3+8bc^4+6b^3c-6bc^3-b^2c+bc^2\right)}{960b\left(2b^2-3b+1\right)\left(2bc^2-2c^3-3bc+3c^2+b-c\right)c}$$

Evaluate (9) at $x=x_{n+1}$ (i.e at $x=1$) yields

$$y_{n+1}=-y_n+2y_{n+\frac{1}{2}}+h^2[\frac{f_n\,(20bc-3)}{960bc}+\frac{f_{n+c}}{(640c-320)\,c\,(b-c)\,(c-1)}+$$

$$\frac{f_{n+\frac{1}{2}}\,(50bc-25b-25c+14)}{(120b-60)\,(2c-1)}-\frac{f_{n+b}}{(640b-640c)\,b\,(b-1/2)\,(b-1)}+\frac{f_{n+1}\,(20bc-20b-20c+17)}{(960b-960)\,(c-1)}] \tag{10}$$

rewriting (10) gives

$$y_{n+1}+y_n-2y_{n+\frac{1}{2}}-h^2[\frac{f_n\,(20bc-3)}{960bc}+\frac{f_{n+c}}{(640c-320)\,c\,(b-c)\,(c-1)}+$$
$$\frac{f_{n+\frac{1}{2}}\,(50bc-25b-25c+14)}{(120b-60)\,(2c-1)}-$$

$$\frac{f_{n+b}}{(640b-640c)\,b\left(b-\frac{1}{2}\right)(b-1)}+\frac{f_{n+1}\,(20bc-20b-20c+17)}{(960b-960)\,(c-1)}]=0 \tag{11}$$

Expanding each terms in (11) using Taylor series expansion gives:

$$y_{n+1}=\frac{h^0}{0!}y_n+\frac{h}{1!}y_n'+\frac{h^2}{2!}y_n''+\frac{h^3}{3!}y_n'''+\frac{h^4}{4!}y_n''''+\frac{h^5}{5!}y_n'''''++\frac{h^6}{6!}y_n''''''+\cdots$$

$$y_n=[y_n]$$

$$-2y_{n+\frac{1}{2}}=-2\left[\frac{h^0}{0!}y_n+\frac{\frac{1}{2}h}{1!}y_n'+\frac{(\frac{1}{2}h)^2}{2!}y_n''+\frac{(\frac{1}{2}h)^3}{3!}y_n'''+\frac{(\frac{1}{2}h)^4}{4!}y_n''''+\frac{(\frac{1}{2}h)^5}{5!}y_n'''''++\frac{(\frac{1}{2}h)^6}{6!}y_n''''''+\cdots\right]$$

$$-\frac{20bc-3}{960bc}f_n=-\left[\frac{20bc-3}{960bc}\right]y_n''$$

$$-\frac{1}{(640c-320)\,c\,(b-c)\,(c-1)}f_{n+c}=$$

$$\frac{1}{(640c-320)\,c\,(b-c)\,(c-1)}\left[\frac{(ch)^0}{0!}y_n''+\frac{(ch)^1}{1!}y_n'''+\frac{(ch)^2}{2!}y_n''''+\frac{(ch)^3}{3!}y_n'''''+\frac{(ch)^4}{4!}y_n''''''+\cdots\right]$$

$$-\frac{50bc-25b-25c+14}{(120b-60)\,(2c-1)}f_{n+\frac{1}{2}}=$$

$$-\frac{50bc-25b-25c+14}{(120b-60)\,(2c-1)}\left[\frac{h^0}{0!}y_n''+\frac{(\frac{1}{2}h)^1}{1!}y_n'''+\frac{(\frac{1}{2}h)^2}{2!}y_n''''+\frac{(\frac{1}{2}h)^3}{3!}y_n'''''+\frac{h^4}{4!}y_n''''''+\cdots\right]$$

$$+\frac{1}{(640b-640c)\,b\,(b-1/2)\,(b-1)}f_{n+b}=$$

$$\frac{1}{(640b-640c)\,b\,(b-1/2)\,(b-1)}\left[\frac{(bh)^0}{0!}y_n''+\frac{(bh)^1}{1!}y_n'''+\frac{(bh)^2}{2!}y_n''''+\frac{(bh)^3}{3!}y_n'''''+\frac{(bh)^4}{4!}y_n''''''+\cdots\right]$$

$$-\frac{20bc-20b-20c+17}{(960b-960)\,(c-1)}f_{n+1}=$$

$$-\frac{20bc-20b-20c+17}{(960b-960)(c-1)}\left[\frac{(h)^0}{0!}y_n''+\frac{(h)^1}{1!}y_n'''+\frac{(h)^2}{2!}y_n''''+\frac{(h)^3}{3!}y_n'''''+\frac{(h)^4}{4!}y_n''''''+\cdots\right]$$

Comparing each of the coefficient above , we have:

$$h^0 y_n : 1+1-2=0$$

$$h^1 y_n' : 1-2\frac{1}{2}=0$$

$$h^2 y_n'' : \left(\frac{1}{2!}\right)-2\frac{\left(\frac{1}{2}\right)^2}{2!}-\frac{c^{2-2}}{(640c-320)c(b-c)(c-1)(2-2)!}-\frac{(50bc-25b-25c+14)\left(\frac{1}{2}\right)^{2-2}}{(120b-60)(2c-1)(2-2)!}$$
$$+\frac{b^{2-2}}{(640b-640c)b\left(b-\frac{1}{2}\right)(b-1)(2-2)!}-\frac{20bc-20b-20c+17}{(960b-960)(c-1)(2-2)!}=0$$

$$h^3 y_n''' : \left(\frac{1}{3!}\right)-2\frac{\left(\frac{1}{2}\right)^3}{3!}-\frac{c^{3-2}}{(640c-320)c(b-c)(c-1)(3-2)!}-\frac{(50bc-25b-25c+14)\left(\frac{1}{2}\right)^{3-2}}{(120b-60)(2c-1)(3-2)!}$$
$$+\frac{b^{3-2}}{(640b-640c)b\left(b-\frac{1}{2}\right)(b-1)(3-2)!}-\frac{20bc-20b-20c+17}{(960b-960)(c-1)(3-2)!}=0$$

$$h^4 y_n'''' : \left(\frac{1}{4!}\right)-2\frac{\left(\frac{1}{2}\right)^4}{4!}-\frac{c^{4-2}}{(640c-320)c(b-c)(c-1)(4-2)!}-\frac{(50bc-25b-25c+14)\left(\frac{1}{2}\right)^{4-2}}{(120b-60)(2c-1)(4-2)!}$$
$$+\frac{b^{4-2}}{(640b-640c)b\left(b-\frac{1}{2}\right)(b-1)(4-2)!}-\frac{20bc-20b-20c+17}{(960b-960)(c-1)(4-2)!}=0$$

$$h^5 y_n''''' : \left(\frac{1}{5!}\right)-2\frac{\left(\frac{1}{2}\right)^5}{5!}-\frac{c^{5-2}}{(640c-320)c(b-c)(c-1)(5-2)!}-\frac{(50bc-25b-25c+14)\left(\frac{1}{2}\right)^{5-2}}{(120b-60)(2c-1)(5-2)!}$$
$$+\frac{b^{5-2}}{(640b-640c)b\left(b-\frac{1}{2}\right)(b-1)(5-2)!}-\frac{20bc-20b-20c+17}{(960b-960)(c-1)(5-2)!}=0$$

$$h^6 y_n'''''' : \left(\frac{1}{6!}\right)-2\frac{\left(\frac{1}{2}\right)^6}{6!}-\frac{c^{6-2}}{(640c-320)c(b-c)(c-1)(6-2)!}-\frac{(50bc-25b-25c+14)\left(\frac{1}{2}\right)^{6-2}}{(120b-60)(2c-1)(6-2)!}$$
$$+\frac{b^{6-2}}{(640b-640c)b\left(b-\frac{1}{2}\right)(b-1)(6-2)!}-\frac{20bc-20b-20c+17}{(960b-960)(c-1)(6-2)!}=0$$

$$h^7 y_n''''''' : \left(\frac{1}{7!}\right)-2\frac{\left(\frac{1}{2}\right)^7}{7!}-\frac{c^{7-2}}{(640c-320)c(b-c)(c-1)(7-2)!}-\frac{(50bc-25b-25c+14)\left(\frac{1}{2}\right)^{7-2}}{(120b-60)(2c-1)(7-2)!}$$
$$+\frac{b^{7-2}}{(640b-640c)b\left(b-\frac{1}{2}\right)(b-1)(7-2)!}-\frac{20bc-20b-20c+17}{(960b-960)(c-1)(7-2)!}\neq 0$$

$$L[y(x_n),h]=\frac{b}{76800}+\frac{c}{76800}-\frac{1}{76800}y_n h^7+O(h^8) \tag{12}$$

To include the initial condition in equation (1), we can take the derivative of the continuous method (9) with respect to x

$$hy'(x)=\alpha_0' y_n+\alpha_b' y_{n+\frac{1}{2}}+h^2\left[\beta_0' f_n+\beta_c' f_{n+c}+\beta_{\frac{1}{2}}' f_{n+\frac{1}{2}}+\beta_b' f_{n+b}+\beta_1' f_{n+1}\right], \tag{13}$$

Evaluate (13) at $x=x_{n+1}$ (i.e at $x=1$) yields

$$hy_{n+1}'=-2y_n+2y_{n+\frac{1}{2}}+h^2[\frac{(20bc+16b+16c-19)f_n}{960bc}-\frac{(16b-19)f_{n+c}}{(1920c-960)(b-c)c(c-1)}+$$
$$\frac{(130bc-73b-73c+46)f_{n+\frac{1}{2}}}{(120c-60)(2b-1)}+\frac{(16c-19)f_{n+b}}{(960b-960)(2b-1)b(b-c)}$$
$$+\frac{(180bc-164b-164c+145)f_{n+1}}{(960b-960)(c-1)}] \tag{14}$$

In a similar fashion as equation (12) was obtained from equation (10), we obtain the local truncation error for equation (14) as

$$L[y(x_n),h]=\frac{(-112c+133)b}{1612800}+\frac{19c}{230400}-\frac{13}{179200}y_n h^7+O(h^8) \tag{15}$$

By forcing the leading terms in equations (12) and (15) to equal zero, we obtain the following system of nonlinear equations.

$$\frac{(-112c+133)b}{1612800}+\frac{19c}{230400}-\frac{13}{179200}=0$$

$$\frac{b}{76800}+\frac{c}{76800}-\frac{1}{76800}=0,$$

Because the implicit system describes a curve symmetrical around the diagonal, we can expect a unique solution for a given set of constraints. i.e. $0<c<\frac{1}{2}<b<1$

$$c=\frac{1}{2}-\frac{1}{14}\sqrt{21}$$
$$b=\frac{1}{2}+\frac{1}{14}\sqrt{21}$$

Substituting the value of c and b into equation (10) gives:

$$y_{n+1}=-y_n+2y_{n+\frac{1}{2}}+h^2\left(-\frac{f_n}{960}+\frac{49\,f_{n+\frac{1}{2}-\frac{1}{14}\sqrt{21}}}{960}+\frac{3\,f_{n+\frac{1}{2}}}{20}+\frac{49\,f_{n+\frac{1}{2}-\frac{1}{14}\sqrt{21}}}{960}-\frac{f_{n+1}}{960}\right) \tag{16}$$

Also evaluating (9) at $x=x_{n+c}$ and $x=x_{n+b}$ gives

$$y_{n+\frac{1}{2}-\frac{1}{14}\sqrt{21}}=-\frac{1}{7}\sqrt{21}y_n+\left(\frac{1}{7}\sqrt{21}+1\right)y_{n+\frac{1}{2}}+\left(-\frac{h^2\sqrt{21}}{13440}-\frac{27\,h^2}{31360}\right)f_n+$$
$$\left(\frac{7\,h^2\sqrt{21}}{1920}+\frac{247\,h^2}{13440}\right)f_{n+\frac{1}{2}-\frac{1}{4}\sqrt{21}}+\left(\frac{3\,h^2\sqrt{21}}{280}+\frac{83\,h^2}{1960}\right)f_{n+\frac{1}{2}}+\left(\frac{7\,h^2\sqrt{21}}{1920}-\frac{73\,h^2}{13440}\right)f_{n+\frac{1}{2}+\frac{1}{14}\sqrt{21}}$$
$$+\left(-\frac{h^2\sqrt{21}}{13440}-\frac{27\,h^2}{31360}\right)f_{n+1} \tag{17}$$

$$y_{n+\frac{1}{2}+\frac{1}{14}\sqrt{21}}=-\frac{1}{7}\sqrt{21}y_n+\left(\frac{1}{7}\sqrt{21}+1\right)y_{n+\frac{1}{2}}+\left(-\frac{h^2\sqrt{21}}{13440}-\frac{27\,h^2}{31360}\right)f_n+$$
$$\left(\frac{7\,h^2\sqrt{21}}{1920}+\frac{247\,h^2}{13440}\right)f_{n+\frac{1}{2}-\frac{1}{14}\sqrt{21}}+\left(\frac{3\,h^2\sqrt{21}}{280}+\frac{83\,h^2}{1960}\right)f_{n+\frac{1}{2}}+\left(\frac{7\,h^2\sqrt{21}}{1920}-\frac{73\,h^2}{13440}\right)f_{n+\frac{1}{2}+\frac{1}{14}\sqrt{21}}$$
$$+\left(-\frac{h^2\sqrt{21}}{13440}-\frac{27\,h^2}{31360}\right)f_{n+1} \tag{18}$$

Substituting the value of c and b into (14) gives:

$$hy'_{n+1}=-2y_n+2y_{n+\frac{1}{2}}-\frac{f_nh^2}{960}+\left(-\frac{7\sqrt{21}}{360}+\frac{539}{2880}\right)h^2f_{n+\frac{1}{2}-\frac{1}{14}\sqrt{21}}+\frac{59\,f_{n+\frac{1}{2}}h^2}{180}+$$
$$\left(\frac{7\sqrt{21}}{360}+\frac{539}{2880}\right)h^2f_{n+\frac{1}{2}+\frac{1}{14}\sqrt{21}}+\frac{47h^2\,f_{n+1}}{960} \tag{19}$$

Evaluate (13) at $x=x_n$ (i. e $x=0$) ,$x=x_{n+c}$ (i.e $x=c$), $x=x_{n+\frac{1}{2}}$ (i. e $x=\frac{1}{2}$), $x=x_{n+b}$ (i. e $x=b$),

$$hy'_n=-2y_n+2y_{n+\frac{1}{2}}-\frac{49\,f_nh^2}{960}+\left(-\frac{7\sqrt{21}}{360}-\frac{49}{576}\right)h^2f_{n+\frac{1}{2}-\frac{1}{14}\sqrt{21}}-\frac{1}{36}f_{n+\frac{1}{2}}h^2+$$
$$\left(\frac{7\sqrt{21}}{360}-\frac{49}{576}\right)h^2f_{n+\frac{1}{2}+\frac{1}{14}\sqrt{21}}-\frac{f_{n+1}h^2}{960} \tag{20}$$

$$hy'_{n+\frac{1}{2}-\frac{1}{14}\sqrt{21}}=-2y_n+2y_{n+\frac{1}{2}}+\left(\frac{3\sqrt{21}}{1960}+\frac{13}{1344}\right)h^2f_n+\left(-\frac{29\sqrt{21}}{1260}+\frac{49}{960}\right)h^2f_{n+\frac{1}{2}-\frac{1}{14}\sqrt{21}}+$$
$$\left(-\frac{32\sqrt{21}}{735}+\frac{3}{20}\right)h^2f_{n+\frac{1}{2}}+\left(-\frac{h^2\sqrt{21}}{126}+\frac{49\,h^2}{960}\right)h^2f_{n+\frac{1}{2}+\frac{1}{14}\sqrt{21}}+\left(\frac{3\,h^2\sqrt{21}}{1960}-\frac{79\,h^2}{6720}\right)h^2f_{n+1} \tag{21}$$

$$hy'_{n+\frac{1}{2}}=-2y_n+2y_{n+\frac{1}{2}}-\frac{f_nh^2}{96}+\left(\frac{49\sqrt{21}}{2880}+\frac{49}{960}\right)h^2f_{n+\frac{1}{2}-\frac{1}{14}\sqrt{21}}+\frac{3\,f_{n+\frac{1}{2}}h^2}{20}+$$
$$\left(-\frac{49\sqrt{21}}{2880}+\frac{49}{960}\right)h^2f_{n+\frac{1}{2}+\frac{1}{14}\sqrt{21}}+\frac{f_{n+1}h^2}{120} \tag{22}$$

$$hy'_{n+\frac{1}{2}+\frac{1}{14}\sqrt{21}}=-2y_n+2y_{n+\frac{1}{2}}+\left(-\frac{3\sqrt{21}}{1960}+\frac{13}{1344}\right)h^2f_n+\left(\frac{\sqrt{21}}{126}+\frac{49}{960}\right)h^2f_{n+\frac{1}{2}-\frac{1}{14}\sqrt{21}}+$$
$$\left(\frac{32\sqrt{21}}{735}+\frac{3}{20}\right)h^2f_{n+\frac{1}{2}}+\left(\frac{29\sqrt{21}}{1260}+\frac{49}{960}\right)h^2f_{n+\frac{1}{2}+\frac{1}{14}\sqrt{21}}+\left(-\frac{3\sqrt{21}}{1960}-\frac{79}{6720}\right)h^2f_{n+1} \tag{23}$$

Equation (16–23) will be combine as a block, translate into Computer code for solving numerical problems in section 4.

3. Analysis of the derived method

The order of the method, consistency, zero stability, convergence and region of absolute stability measure the basic properties of the method are considered in this section.

3.1. Order and Error Constant

The order and error term of the derived method is found by defining a linear operator L as:

$$L(y(x_n),h) = \sum_{j=0}^{k} \left\{ \alpha_j y(x_{n+j}) - h^2 \delta_j y''(x_{n+j}) \right\} \tag{24}$$

where $\alpha_k = 1$, α_0 and δ_0 are not both not zero, $y(x)$ is an arbitrary function which is continuously differentiable on the interval $[a,z]$, $yx_{n+j} = y(x_n + jh)$.
If $y(x)$ represent the true solution of (1) and we adopt Taylor series expansion of $y(x_{n+j})$ and $y''(x_{n+j})$ about $x = x_n$ [2, 8].

Table 1: Order and Error Constants of the methods

Method Eqn. No.	Order	Error constant
(16)	6	$\frac{1}{15482880}$
(17)	6	$-\frac{8913}{74348789760}\sqrt{21}$
(18)	6	$-\frac{31}{268933120}\sqrt{21}$
(19)	6	$\frac{1}{7546873}$
(20)	6	$\frac{1}{15482880}$
(21)	6	$-\frac{79}{80676400}$
(22)	6	$\frac{1}{7856740}$
(23)	6	$\frac{77}{88565974}$

3.2. Consistency of the Method

According to [4, 23] . A block method is said to be consistent, if it has an order of convergence, say $\rho \geq 1$ Our method is consistent as all our method are of order $p > 1$.

3.3. Zero Stability

The block method is zero stable if the roots z_s of the first characteristic polynomial $\rho(z)$ which is defined as

$$\bar{\rho}(z) = \det[zI_n - \bar{E}] \tag{25}$$

satisfy $|z_s| \leq 1$ and every root with $|z_s| = 1$ has multiplicity not exceeding in the limit as $h \longrightarrow 1$ and I_5 is an identity matrix.

$$\bar{E} = \begin{pmatrix} -1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{7}\sqrt{21} & \frac{1}{7}\sqrt{21}+1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{7}\sqrt{21} & \frac{1}{7}\sqrt{21}+1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Thus

$$\bar{\rho}(z) = det[zI_5 - \bar{E}] = z^8 - \frac{1}{7}z^7\sqrt{21} + \frac{1}{7}\sqrt{21}z^6 - z^6$$

whose zeros are $z = -1$ or $z = 0$ or $z = \frac{1}{7}\sqrt{21} - 1$, hence the block method is zero stable.

3.4. Convergence of the Method

[Fundamental Theorem of Dahlquist[24]] *The necessary and sufficient condition for a linear multistep to be convergent is for it to be consistent and zero stable.*
The block method are Convergent since consistent + zero stability ≡ Convergence

3.5. Region of Absolute Stability the Method

According to [20, 21], the stability matrix is defined as

$$M(z) = V + zB(I - ZA)^{-1}U \tag{26}$$

$$P(\eta, z) = \det(\eta I - M(z)) \tag{27}$$

$$\begin{bmatrix} y_n \\ \\ -- \\ y_{n+i} \end{bmatrix} = \begin{bmatrix} A & U \\ \\ --- & --- \\ C & V \end{bmatrix} \begin{bmatrix} h^2 f(y) \\ \\ \\ y_{n-i} \end{bmatrix}$$

where
U and V are obtained from the interpolation points while A and C are the coefficient of the collocation points. The element of matrix A,C,U and V are then used in (26) and (27). we solve the resulting equation using maple sofware to obtain the stability polynomial of the method respectively. The stability polynomial are then plotted using MATLAB to generate the stability region of the methods

$$C = \begin{pmatrix}
-\frac{1}{960} & \frac{49}{960} & \frac{3}{20} & \frac{49}{960} & -\frac{1}{960} & 0 & 0 & 0 \\
-\left(\frac{\sqrt{21}}{13440} - \frac{27}{31360}\right) & \left(\frac{7\sqrt{21}}{13440} - \frac{247}{13440}\right) & \left(\frac{3\sqrt{21}}{280} + \frac{83}{1960}\right) & 7\left(\frac{\sqrt{21}}{1920} - \frac{73}{13440}\right) & \left(-\frac{\sqrt{21}}{13440} - \frac{27}{31360}\right) & 0 & 0 & 0 \\
-\left(\frac{\sqrt{21}}{13440} - \frac{27}{31360}\right) & \left(\frac{7\sqrt{21}}{13440} - \frac{247}{13440}\right) & \left(\frac{3\sqrt{21}}{280} + \frac{83}{1960}\right) & 7\left(\frac{\sqrt{21}}{1920} - \frac{73}{13440}\right) & \left(-\frac{\sqrt{21}}{13440} - \frac{27}{31360}\right) & 0 & 0 & 0 \\
-\frac{1}{90} & \left(-\frac{7\sqrt{21}}{360} + \frac{539}{2880}\right) & \frac{59}{180} & \left(-\frac{7\sqrt{21}}{360} + \frac{539}{2880}\right) & \frac{47}{960} & 0 & 0 & 0 \\
-\frac{1}{90} & \left(-\frac{7\sqrt{21}}{360} - \frac{49}{576}\right) & -\frac{1}{36} & \left(\frac{7\sqrt{21}}{360} - \frac{49}{576}\right) & -\frac{1}{960} & 0 & 0 & 0 \\
\left(\frac{3\sqrt{21}}{1960} + \frac{13}{1344}\right) & \left(\frac{49\sqrt{21}}{1260} + \frac{49}{960}\right) & \left(\frac{32\sqrt{21}}{735} + \frac{3}{200}\right) & \left(-\frac{\sqrt{21}}{126} + \frac{49}{960}\right) & \left(\frac{3\sqrt{21}}{1960} - \frac{79}{6720}\right) & 0 & 0 & 0 \\
-\frac{1}{96} & \left(49\frac{\sqrt{21}}{2880} + \frac{49}{960}\right) & \frac{3}{20} & \left(49\frac{3\sqrt{21}}{2880} + \frac{49}{960}\right) & \frac{1}{120} & 0 & 0 & 0 \\
\left(\frac{-3\sqrt{21}}{1960} + \frac{49}{1344}\right) & \left(\frac{\sqrt{21}}{126} + \frac{49}{960}\right) & \left(\frac{32\sqrt{21}}{735} + \frac{3}{20}\right) & \left(\frac{29\sqrt{21}}{1260} + \frac{49}{960}\right) & \left(\frac{-3\sqrt{21}}{1960} - \frac{79}{6720}\right) & 0 & 0 & 0
\end{pmatrix}$$

$$A = \begin{pmatrix}
1 & 1 & 2 & 0 & 0 & 0 & 0 & 0 \\
1 & \frac{1}{7}\sqrt{21} & -(\frac{1}{7}\sqrt{21}+1) & 0 & 0 & 0 & 0 & 0 \\
1 & \frac{1}{7}\sqrt{21} & -(\frac{1}{7}\sqrt{21}+1) & 0 & 0 & 0 & 0 & 0 \\
0 & 2 & -2 & 0 & 0 & 0 & 0 & 0 \\
0 & 2 & -2 & 0 & 0 & 0 & 0 & 0 \\
0 & 2 & -2 & 0 & 0 & 0 & 0 & 0 \\
0 & 2 & -2 & 0 & 0 & 0 & 0 & 0 \\
0 & 2 & -2 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}$$

$$V = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}$$

$$U = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}$$

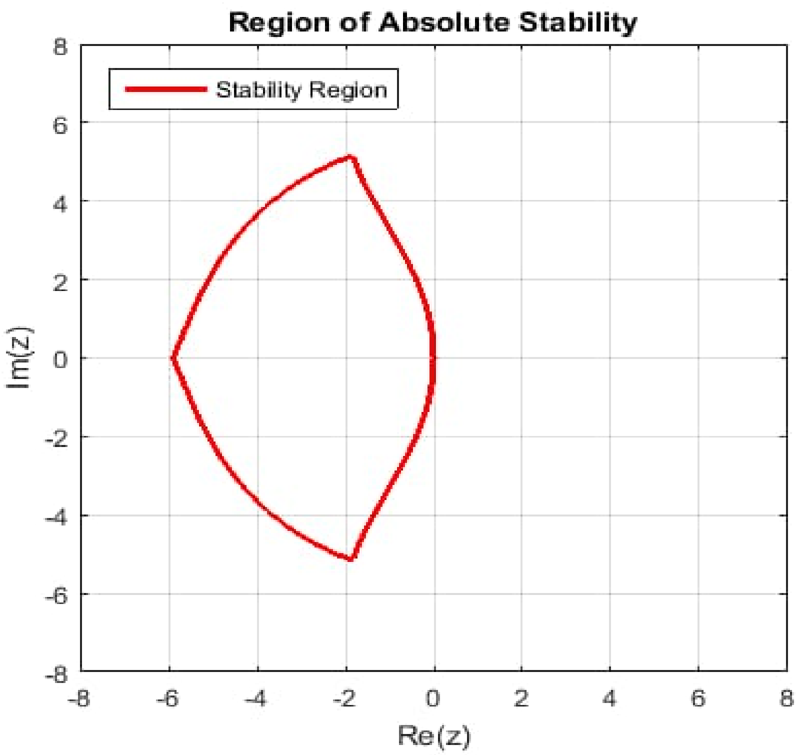

Figure 1: Plot of region of absolute stability for the new method

4. Numerical Illustration

This section outlines the implementation strategy for the proposed hybrid one step block method and evaluates its performance across a series of numerical test cases. To ensure a rigorous assessment, we apply the scheme to a variety of second-order initial value problems (IVPs), ranging from linear to highly nonlinear differential equations. The computational algorithms were developed and executed using the Maple 2018 algebraic environment. The implementation follows a systematic work flow: the continuous formulation is transformed into a discrete block system, which is then solved iteratively or directly to generate approximations at the designated grid and off-grid points.
To quantify the efficiency and precision of the developed method, we compare our numerical results against both exact solutions and existing methods from literature. The findings are presented in tabular form using the following notations:
Y_{EX}: The analytical (exact) solution of the IVP.
Y_N: The numerical approximation generated by the proposed method.
E_R: The absolute error, defined as $|Y_{EX} - Y_N|$.
The following examples demonstrate the robustness of our $O(h^9)$ scheme in maintaining high fidelity across various dynamic behaviors.

Example 1.1

We consider a moderately stiff problem

$$y''(x) = y', \;\; y(x) = 0, \;\; y'(x) = -1$$

Exact solution: $y(x) = 1 - e^x$

Table 2: Result for Example 1.1, with $h = 0.1$

x	Y_{EX}	Y_N	E_R
0.1	−0.1051709180756476248	−0.10517091807564758129	4.351×10^{-17}
0.2	−0.2214027581601698339	−0.22140275816016973772	9.618×10^{-17}
0.3	−0.3498588075760031040	−0.34985880757600294451	1.5949×10^{-16}
0.4	−0.4918246976412703178	−0.49182469764127008283	2.3497×10^{-16}
0.5	−0.6487212707001281468	−0.64872127070012782221	3.2459×10^{-16}
0.6	−0.8221188003905089749	−0.82211880039050854434	4.3056×10^{-16}
0.7	−1.0137527074704765216	−1.0137527074704759665	5.551×10^{-16}
0.8	−1.2255409284924676046	−1.2255409284924669034	7.012×10^{-16}
0.9	−1.4596031111569496638	−1.4596031111569487920	8.718×10^{-16}
1.0	−1.7182818284590452354	−1.7182818284590441648	1.0706×10^{-15}

Table 3: Comparison of Errors for example 1.1

x	AA_{err}[25]	$Ade1_{err}$[7]	OJ_{err}[15]	new method
0.1	8.4×10^{-8}	2.22×10^{-8}	1.53×10^{-8}	4.35×10^{-17}
0.2	5.3×10^{-6}	1.25×10^{-7}	3.39×10^{-8}	9.62×10^{-17}
0.3	6.2×10^{-6}	3.25×10^{-7}	5.63×10^{-8}	1.60×10^{-16}
0.4	1.6×10^{-6}	6.42×10^{-7}	8.29×10^{-8}	2.35×10^{-16}
0.5	1.0×10^{-5}	1.10×10^{-7}	1.15×10^{-7}	3.25×10^{-16}
0.6	2.9×10^{-5}	1.72×10^{-6}	1.52×10^{-7}	4.31×10^{-16}
0.7	5.9×10^{-5}	2.54×10^{-6}	1.96×10^{-7}	5.55×10^{-16}
0.8	1.0×10^{-4}	3.58×10^{-6}	2.47×10^{-7}	7.01×10^{-8}
0.9	1.5×10^{-4}	4.90×10^{-6}	3.07×10^{-7}	8.72×10^{-16}
1.0	2.3×10^{-2}	6.52×10^{-6}	3.78×10^{-7}	1.07×10^{-15}

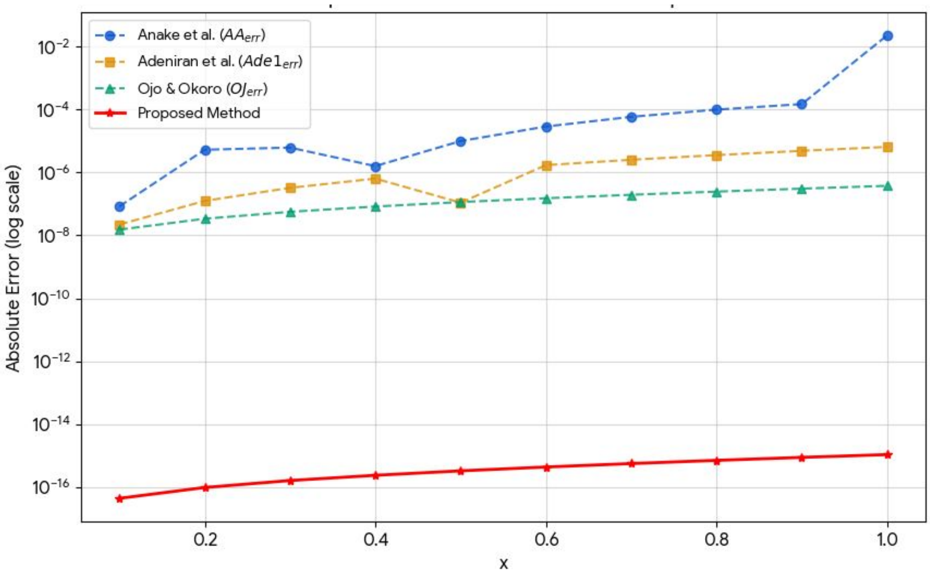

Figure 2: Comparison of Absolute error for Example 1.1

The numerical experiments for Example 1.1 were conducted to evaluate the performance of the proposed method in comparison to existing schemes in the literature.

Table 1 displays the exact solution (Y_{EX}), the computed numerical solution (Y_N), and the absolute error (E_R) at various grid points with a step size of $h = 0.1$. It is evident that the computed solutions are in high agreement with the exact solutions, maintaining several decimal places of accuracy across the entire interval $[0.1, 1.0]$. The absolute errors range from 4.351×10^{-17} at $x = 0.1$ to 1.0706×10^{-15} at $x = 1.0$.

Table 2 provides a comparative study of the absolute errors between the new method and those developed by [25] (AA), [7] (Ade1) and [15] (OJ). The results clearly show that the new method significantly outperforms the existing schemes. While the errors for the existing methods range between 10^{-2} and 10^{-8}, the new method consistently produces errors in the magnitude of 10^{-15} to 10^{-17}. While the absolute error in the new method increases slightly as x approaches 1.0 (a common phenomenon in multistep methods due to error accumulation), the rate of growth is remarkably slow. The method maintains a precision of 15 decimal places even at the end of the interval, whereas the other methods lose significant precision.

Example 1.2

We consider a highly stiff initial value problem

$$y''(x) = -1001y' - 1000y, \quad y(0) = 0, \quad y'(0) = -1$$

Exact solution: $y(x) = e^{-x}$

The numerical results presented in the Table 4 provide a comparative analysis between the proposed method and established methods in literature for solving Example 1.2.

The performance of the proposed method (Y_N) is evaluated against the exact solution (Y_{EX}) over the interval $[0.05, 0.50]$ with a step size of $h = 0.05$.

The absolute error E_R consistently remains within the range of 10^{-17}. Specifically, the error starts at 2.046×10^{-17} at $x = 0.05$ and maintains a stable profile, peaking slightly at 3.146×10^{-17} before descending to 2.452×10^{-17} at the final point. The narrow range of the error values suggests that the method is numerically stable and does not suffer from significant error accumulation as x increases.

The results clearly demonstrate that the proposed numerical scheme is highly efficient. By achieving an absolute error in the order of 10^{-17}, it outperforms the existing methods of [15, 25] by several orders of magnitude. This suggests that the proposed approach provides a much closer approximation to the exact solution, making it a more reliable tool for solving problems of this nature.

Table 4: Result for Example 1.2, with $h = 0.05$

x	Y_{EX}	Y_N	E_R	OJ_{err} [15]	AA_{err}[25]
0.05	0.95122942450071400909	0.95122942450071402955	2.046×10^{-17}	–	–
0.1	0.90483741803595957316	0.90483741803595960187	2.871×10^{-17}	1.04×10^{-10}	1.09×10^{-10}
0.15	0.86070797642505780723	0.86070797642505783869	3.146×10^{-17}	–	–
0.2	0.81873075307798185867	0.81873075307798189045	3.103×10^{-17}	2.04×10^{-10}	2.08×10^{-10}
0.25	0.77880078307140486825	0.77880078307140489928	3.103×10^{-17}	–	–
0.30	0.74081822068171786607	0.74081822068171789592	2.985×10^{-17}	2.99×10^{-10}	3.86×10^{-10}
0.35	0.70468808971871343435	0.70468808971871346287	2.852×10^{-17}	–	–
0.40	0.67032004603563930074	0.67032004603563932789	2.715×10^{-17}	3.87×10^{-10}	3.48×10^{-08}
0.45	0.63762815162177329314	0.63762815162177331895	2.581×10^{-17}	–	–
0.50	0.60653065971263342360	0.60653065971263344812	2.452×10^{-17}	4.68×10^{-09}	3.96×10^{-07}

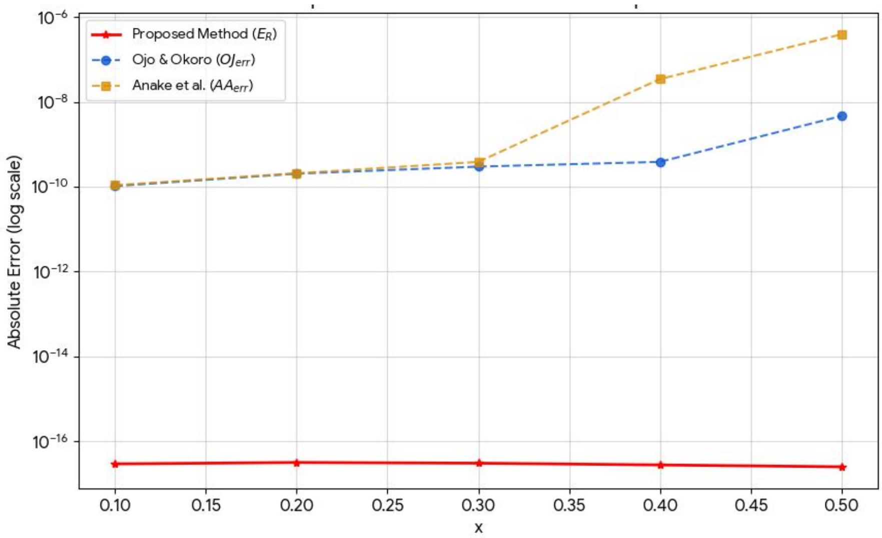

Figure 3: Comparison of Absolute error for Example 1.2

Example 1.3

We consider the initial value problem

$$y''(x) = 100x + 99sin(x), \quad y(0) = 1, \quad y'(0) = 11$$

Exact solution: $y(x) = \cos(10x) + \sin(10x) + \sin(x)$

Table 5: Result for Example 1.3, with $h = \frac{1}{320}$

x	Y_{EX}	Y_N	E_R	$Ade1_{err}$[7]	$Ade2_{err}$[4]
$\frac{1}{320}$	1.0338816673842019107	1.0338816673842019106	1.0×10^{-19}	2.00×10^{-14}	9.17×10^{-11}
$\frac{2}{320}$	1.0667567878524546543	1.0667567878524546542	1.0×10^{-19}	6.00×10^{-14}	–
$\frac{3}{320}$	1.0985962803650165721	1.0985962803650165719	2.0×10^{-19}	1.30×10^{-13}	3.09×10^{-10}
$\frac{4}{320}$	1.1293720750962665318	1.1293720750962665316	2.0×10^{-19}	2.10×10^{-13}	–
$\frac{5}{320}$	1.1590571408149113575	1.1590571408149113574	1.0×10^{-19}	3.60×10^{-13}	–
$\frac{6}{320}$	1.1876255112500243853	1.1876255112500243852	1.0×10^{-19}	5.40×10^{-13}	4.90×10^{-10}
$\frac{7}{320}$	1.2150523104171694481	1.2150523104171694481	0.00	7.40×10^{-13}	–
$\frac{8}{320}$	1.2413137768798800444	1.2413137768798800443	1.0×10^{-19}	9.60×10^{-13}	–
$\frac{9}{320}$	1.2663872869228030575	1.2663872869228030573	2.0×10^{-19}	1.24×10^{-12}	–
$\frac{10}{320}$	1.2902513766138791315	1.2902513766138791312	3.0×10^{-19}	1.55×10^{-12}	–

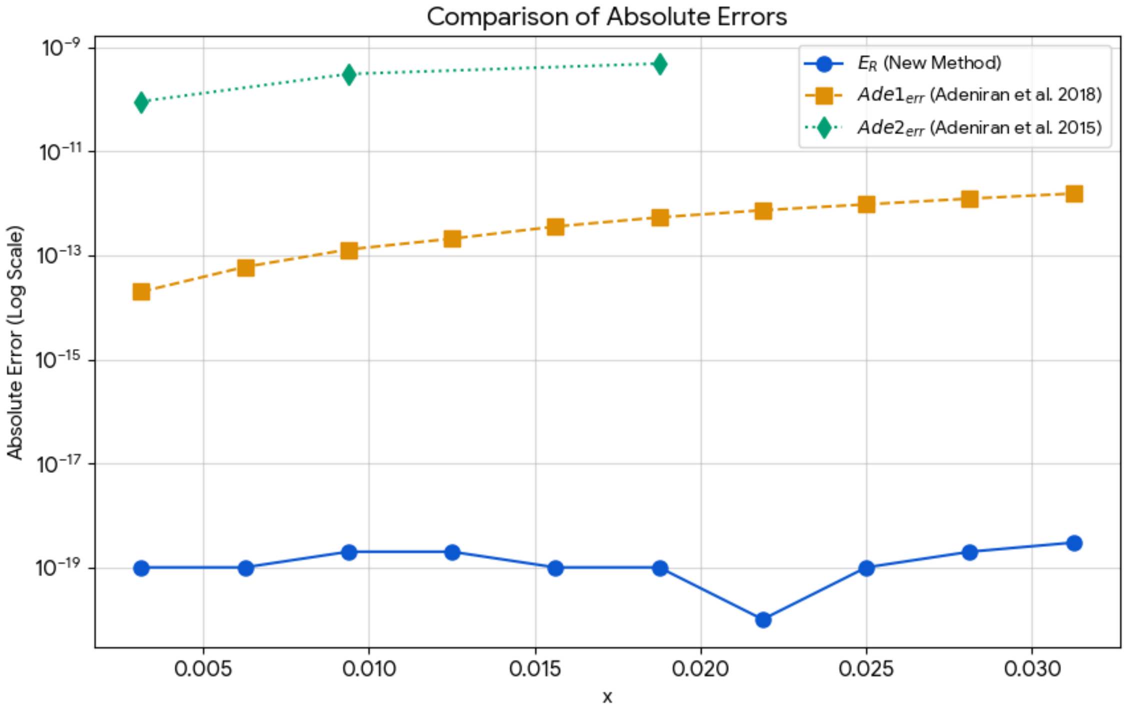

Figure 4: Comparison of Absolute error for Example 1.3

The numerical results presented in the Table 5 demonstrate the high level of accuracy and efficiency of the proposed method.

The proposed method (Y_N) shows a remarkable agreement with the exact solution (Y_{EX}). Throughout the interval $[\frac{1}{320}, \frac{10}{320}]$, the absolute error ($E_R$) remains consistently low, ranging from a minimum of 0.00 at $x = \frac{7}{320}$ to a maximum of only 3.0×10^{-19} at $x = \frac{10}{320}$. This suggests that the method is highly stable and maintains significant precision even as the number of steps increases.

The proposed method consistently outperforms [7]. While [7] produces errors in the magnitude of 10^{-14} to 10^{-12}, the proposed method achieves an accuracy level of 10^{-19}. This represents an improvement of approximately seven orders of magnitude. The disparity is even more pronounced when compared to that of [4]. The errors reported in [4] are significantly higher (reaching 10^{-11} in some instances), whereas the proposed method remains near-zero.

The results confirm that the proposed numerical scheme is superior in terms of precision and reliability for solving Example 1.3, providing a much closer approximation to the exact solution than the previously established methods in the literature.

Example 1.4

We consider the system

$$y_1''(x) = -y_1 + 0.001\cos(x), \;\; y_1(0) = 1, \;\; y_1'(0) = 0$$

$$y_2''(x) = -y_2 + 0.001\sin(x), \;\; y_[2](0) = 0, \;\; y_2'(0) = 0.995$$

Exact solution: $\quad y_1(x) = \cos(x) + 0.0005x\sin(x) \qquad y_2(x) = \sin(x) + 0.0005x\cos(x)$

Table 6: Result for Example 1.4, with h=$\frac{1}{320}$

x	Y_{EX}	Y_N	E_R
$\frac{1}{320}$	0.99999512207427819441	0.99999512207427819441	0.00
$\frac{2}{320}$	0.99998048834470104865	0.99998048834470104865	0.00
$\frac{3}{320}$	0.99995609895403291149	0.99995609895403291149	0.00
$\frac{4}{320}$	0.99992195414021281668	0.99992195414021281668	0.00
$\frac{5}{320}$	0.99987805423635216164	0.99987805423635216164	0.00
x	Z_{EX}	Z_N	E_R
$\frac{1}{320}$	0.0031234324213688510154	0.0031234324213688510154	0.00
$\frac{2}{320}$	0.0062468343710102636872	0.0062468343710102636872	0.00
$\frac{3}{320}$	0.0093701753774940768711	0.0093701753774940768711	0.00
$\frac{4}{320}$	0.012493424969984680920	0.012493424969984680920	0.00
$\frac{5}{320}$	0.015616552678538286186	0.015616552678538286186	0.00

The results presented in the table 6 for Example 1.4 evaluate the performance of the proposed numerical method by comparing the exact solutions (Y_{EX} and Z_{EX}) with the computed numerical solutions (Y_N and Z_N) at a step size of $h = \frac{1}{320}$

The most striking observation from the data is the absolute precision achieved by the proposed method. For both variables, Y and Z, the absolute error (E_R) is consistently recorded as 0.00 across all evaluated points from $x = \frac{1}{320}$ to $x = \frac{5}{320}$. This indicates that the numerical solution coincides with the exact solution to at least 20 decimal places.

The proposed method demonstrates an exceptional degree of accuracy for Example 1.4. The results suggest that the algorithm is capable of producing results that are indistinguishable from the exact solution within the provided precision, confirming the mathematical robustness of the scheme.

Example 1.5

We consider the non linear IVP

$$y_1''(x) = x(y')^2, \ \ y(0) = 1, \ \ y'(0) = \frac{1}{2}$$

Exact solution: $y(x) = 1 + \frac{1}{2} ln\left(\frac{2+x}{2-x}\right)$

Table 7: Result for Example 1.5, with h=0.1

x	Y_{EX}	Y_N	E_R	Awo_{err}[3]	Ram_{err}[18]
0.1	1.0500417292784912682	1.0500417292784914927	$2.24477049 \times 10^{-16}$	6.64×10^{-14}	3.11×10^{-12}
0.2	1.1003353477310755806	1.1003353477310760604	$4.79812273 \times 10^{-16}$	2.00×10^{-09}	6.66×10^{-12}
0.3	1.1511404359364668053	1.1511404359364676118	$8.06556032 \times 10^{-16}$	1.72×10^{-09}	9.83×10^{-12}
0.4	1.2027325540540821910	1.2027325540540834601	$1.269108494 \times 10^{-15}$	5.89×10^{-09}	2.17×10^{-11}
0.5	1.2554128118829953416	1.2554128118829973233	$1.981665093 \times 10^{-15}$	1.44×10^{-08}	3.57×10^{-11}
0.6	1.3095196042031117155	1.3095196042031148769	$3.161447267 \times 10^{-15}$	4.19×10^{-08}	4.86×10^{-11}
0.7	1.3654437542713961691	1.3654437542714014150	$5.245926758 \times 10^{-15}$	5.31×10^{-09}	1.31×10^{-10}
0.8	1.4236489301936018068	1.4236489301936109745	$9.167689294 \times 10^{-15}$	9.11×10^{-08}	2.31×10^{-10}
0.9	1.4847002785940517416	1.4847002785940687852	$1.7043655380 \times 10^{-14}$	1.49×10^{-07}	3.29×10^{-10}
1.0	1.5493061443340548457	1.5493061443340888871	$3.4041449707 \times 10^{-14}$	2.37×10^{-07}	1.33×10^{-09}

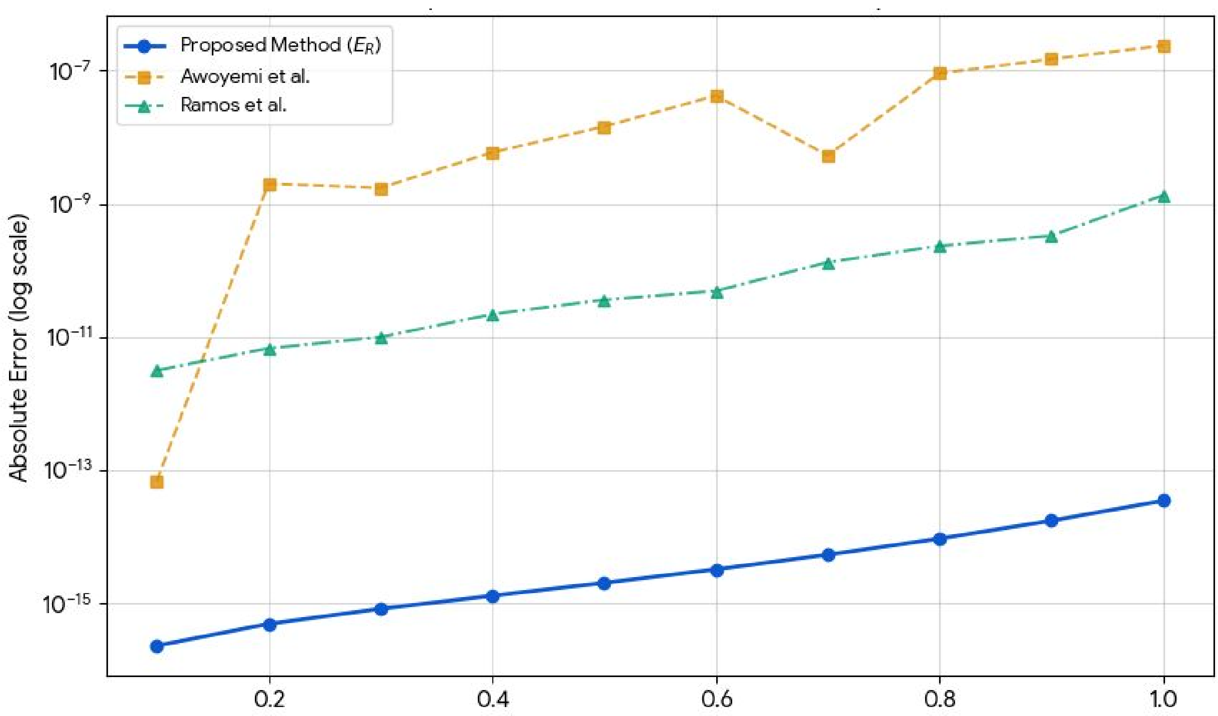

Figure 5: Comparison of Absolute error for Example 1.5

Table 7 displays the numerical results for Example 1.5 using a step size of $h = 0.1$. The performance of the proposed method (Y_N) is evaluated by comparing its absolute error (E_R) against the exact solution (Y_{EX}) and two established methods in the literature: [3, 18].

The proposed method demonstrates exceptional accuracy across the entire interval $[0.1, 1.0]$. The absolute error starts as low as 2.24×10^{-16} at the first step and maintains a high degree of precision, ending at 3.40×10^{-14} at $x = 1.0$. This level of precision indicates that the method is highly effective at capturing the true behavior of the differential equation with minimal divergence.

When compared to the existing methods, the proposed scheme shows a clear competitive advantage. The proposed method is roughly seven orders of magnitude more accurate than the [3]. Similarly, the proposed method outperforms [18] results consistently. The proposed method provides a much tighter approximation to the exact solution throughout the integration process. The empirical evidence from Example 1.5 confirms that the proposed method is superior to the methods of [3] and [18]. in terms of both absolute accuracy and the containment of global truncation error.

5. Conclusion

In this research, we have developed and analyzed an optimized hybrid one-step block method for the direct numerical integration of second-order initial value problems (IVPs). By employing a power series basis function and a strategic collocation-interpolation technique, we successfully bypassed the conventional, computationally expensive requirement of reducing second-order ODEs into systems of first-order equations.

We utilized a 6th-degree power series expansion to derive a continuous formulation of the hybrid method. This approach allows for the evaluation of the solution and its derivative at any point within the computational interval.

A significant highlight of this work is the optimization of the off-grid points. By performing a rigorous Taylor series expansion and analyzing the local truncation error (LTE), we determined the optimal values for the off-grid nodes:

$$c = \frac{1}{2} - \frac{\sqrt{21}}{14} \approx 0.17267 \quad \text{and} \quad b = \frac{1}{2} + \frac{\sqrt{21}}{14} \approx 0.82733$$

The forcing of the leading error terms to zero resulted in a scheme with a high order of accuracy, specifically $O(h^8)$. This ensures rapid convergence and high precision, even with larger step sizes.

The derived discrete schemes and their corresponding derivative formulas (as seen in equations (16) through (23)) provide a self-starting block method that is both robust and computationally efficient for modeling physical phenomena like harmonic motion and electrical circuit dynamics.

The optimized hybrid method presented here offers a powerful alternative to standard numerical integrators. Its ability to provide high-order accuracy while maintaining the structural integrity of the second-order problem makes it highly suitable for complex simulations in Physics, Engineering, and Mathematical Biology. Future work could involve extending this optimization strategy to higher-order differential equations or exploring the stability regions of the method for stiff problems.

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